

**First Project for Math 6644 : Iterative Methods for Systems of Equations**  
(Due on Thursday, March 15 , 2007)

You are encouraged to design your own projects if you are particularly interested in solving some practical problems using the course materials. Please come to see me if you have any idea in your mind.

PRECONDITIONED CONJUGATE GRADIENT METHODS

**Part I.** Discretize one-dimensional equation

$$(1) \quad \begin{cases} -u'' = x - \frac{1}{2} & x \in [0, 1] \\ u(0) = 1, \quad u(1) = 0 \end{cases}$$

by centered difference scheme with  $n$  interior mesh points. Solve the resulting linear system by Conjugate Gradient (CG), Preconditioned CG with Sine transform ( $[S]_{j,k} = \sqrt{\frac{2}{n+1}} \sin(\frac{\pi jk}{n+1})$  for  $1 \leq j, k \leq n$ ) to construct the preconditioner. Run your code with  $n = 16, 32, 64, 128, 256$ . Compare the iteration numbers used for each case in a table. Discuss your own results.

**Hint:**

1. Sine transform is unitary, the forward and inverse transform can be achieved by fast Sine transform or fast inverse Sine transform in  $O(n \log n)$  operations. You can use the fast Sine transform defined in Matlab or any other software packages. However, you should make sure that those functions obtain the correct answers as you want, i.e. in different softwares, the origin corresponds to different index in the Sine transform matrix, you might need to do the re-arrangement of the index if necessary.
2. For this problem, you probably want to consider the splitting preconditioner approach and use the following facts to modify the algorithm to achieve a super fast convergence:
  - (a) You might observe that matrix  $SAS^T$  is diagonal, where  $A$  is the matrix discretized from the differential equation and  $S$  is the Sine transform matrix.
  - (b) The best preconditioner for a diagonal system is its inverse.

**Part II.** A  $n \times n$  matrix  $A_n$  is called Toeplitz matrix if it has the form

$$A_n = \begin{bmatrix} a_0 & a_{-1} & \cdots & a_{2-n} & a_{1-n} \\ a_1 & a_0 & a_{-1} & & a_{2-n} \\ \vdots & a_1 & a_0 & \ddots & \vdots \\ a_{n-2} & & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix},$$

i.e.  $A_n$  is constant along its diagonal. In this part of the project, we consider solving symmetric positive definite Toeplitz systems by preconditioned CG methods using circulant matrices as the preconditioners.

A  $n \times n$  matrix  $C_n$  is called circulant matrix if

$$C_n = \begin{bmatrix} c_0 & c_{-1} & \cdots & c_{2-n} & c_{1-n} \\ c_{1-n} & c_0 & c_{-1} & & c_{2-n} \\ \vdots & c_{1-n} & c_0 & \ddots & \vdots \\ c_{-2} & & \ddots & \ddots & c_{-1} \\ c_{-1} & c_{-2} & \cdots & c_{1-n} & c_0 \end{bmatrix}.$$

Circulant matrices are diagonalized by the Fourier matrix  $F_n$ , i.e.

$$C_n = F_n^* \Lambda_n F_n,$$

where  $[F]_{j,k} = \frac{1}{\sqrt{n}} e^{2\pi i j k / n}$  for  $0 \leq j, k \leq n-1$  and  $\Lambda_n$  is a diagonal matrix holding the eigenvalues of  $C_n$ . In fact,  $\Lambda_n$  can be obtained in  $O(n \log n)$  operations by taking FFT (Fast Fourier Transform) of the first column of  $C_n$  (why?). Once  $\Lambda_n$  is obtained, the products  $C_n \vec{y}$  and  $C_n^{-1} \vec{y}$  can be computed by FFT using  $O(n \log n)$  operations for any given vector  $\vec{y}$ .

Solving a symmetric positive definite Toeplitz system

$$A_n \vec{x} = \vec{b}$$

by CG can be speed up by using circulant preconditioners. Commonly used circulant preconditioners include:

- (i) G. Strang's circulant preconditioner which is defined as  $[C_n]_{k,l} = c_{k-l}$  for  $0 \leq k, l < n$ , where

$$c_j = \begin{cases} a_j & 0 \leq j \leq [n/2] \\ a_{j-n} & [n/2] < j < n \\ c_{n+j} & 0 < -j < n \end{cases},$$

where operator  $[x]$  returns the closest integer (smaller) than  $x$ .

- (ii) T. Chan's circulant preconditioner

$$c_j = \begin{cases} \frac{(n-j)a_j + ja_{j-n}}{n} & 0 \leq j < n \\ c_{n+j} & 0 < -j < n \end{cases}$$

Write your code to perform the CG and PCG for Toeplitz systems using both circulant matrices as the preconditioners. The right hand side  $\vec{b}$  is a random vector select by you.

Experiment your code for the following symmetric toeplitz systems with  $n$  varying from 50, 100, 200, 400:

- (a)  $a_k = |k+1|^{-p}$  for the lower triangular part of  $A_n$ , where  $p = 2, 1, 1/10, 1/100$ . (for  $p \leq 1$ , is the system still positive definite?)  
 (b) The elements of the Toeplitz system is defined by

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta, \quad k = 0, \pm 1, \pm 2, \dots,$$

where  $f(\theta) = \theta^4 + 1$  for  $-\pi \leq \theta \leq \pi$ . This means that  $a_k$  is the Fourier coefficients of function  $f(\theta)$ , and you should obtain  $a_k$  by FFT.

Terminate your computations if the relative residual  $\|\vec{r}_k\|/\|\vec{r}_0\| \leq 10^{-6}$ . List the number of iterations needed for CG without preconditioner, and PCG with G. Strang's and T. Chan's circulant preconditioners in a table in each case. Also compare the number of flops (in case

you can not count the flops easily, use the CPU time instead) against the matrix size  $N$  in each case. Do you get the  $O(n \log n)$  growing pattern? Comment on your results.

**Remark:** Computing  $A_n \vec{x}$  for a Toeplitz system can also be carried out in  $O(n \log n)$  operations by using FFT. A hint is to embed  $A_n$  into a  $2n \times 2n$  circulant matrix and extend  $\vec{x}$  to a  $2n$ -vector by adding zeros. Then by using FFT, one can compute the circulant matrix-vector multiplication and obtain the corresponding  $A_n \vec{x}$  from the resulting  $2n$ -vector.

### CONVERGENCE STUDY FOR PCG

**Part III.** The following part of the project is optional, you are encourage to finish it. You earn 20 bonus points if you have finish this part of the project.

- (a) Exercise 29.1 in the book “*Numerical Linear Algebra*” by L. Trefethen and D. Bau, III.
- (b) Use your code in item (b) to study the eigenvalue distributions of the Toeplitz systems  $A_n$  given in Part II, and compare them with the eigenvalue distributions of preconditioned systems by using the circulant preconditioners. What is your conclusion? Explain how does it affect the convergence of the PCG?

### REFERENCES

- [1] G. Strang, *A Proposal for Toeplitz Matrix Calculations*, Stud. Appl. Math., 74(1986), pp171-176.
- [2] R. Chan and G. Strang, *Toeplitz Equations by Conjugate Gradients with Circulant Preconditioner*, SIAM J. Sci. Stat. Comput., 10(1989), pp 104-117.
- [3] T. Chan, *An Optimal Circulant Preconditioner for Toeplitz Systems*, SIAM J. Sci. Stat. Comput. 9 (1988) pp 766-771.